Structuralism and Category Theory

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1 Introduction

Category theory has proven to be an effective tool for describing a variety of mathematical structures from topological to algebraic to logical ones. Moreover, topos theory can be used to construct enough of set theory within category theory. Thus, some category theorists have claimed that category theory can provide an alternative *conceptual framework for understanding mathematics*. In particular, Steve Awodey [1] suggests that philosophers use category theory in order to clarify the notion of mathematical structure.

On the other hand, Geoffrey Hellman [7], a philosophical structuralist, put some questions to the idea of using category theory as a structuralist framework for mathematics. In replying Hellman's objections, Awodey [2] argues that Hellman, adhering conventional *foundationalism*, overlooks an essential feature of mathematical practice, namely, shcematic and top-down character.

This debate between Awodey and Hellman illustrates well what is at issue in the philosophy of mathematics, and what obstacles lie in the way of establishing a theory of meaning acceptable both philosophically and mathematically.

In the following we shall review the debate between Awodey and Hellman, see what is at issue, and identify what prevents philosophers like Hellman from appreciating what mathematicians like Awodey suggest. Then we shall draw some morals from the debate, and propose a way to interpret mathematical language. It stands in sharp contrast with those considered by philosophers like Hellman in that it is based on a quite anti-foundationalist view of mathematics. Yet it has contituity with at least some part of our ordinary language.

2 The structuralist controversy

One of the major tasks in the philosophy of mathematics is to propose a way of interpreting mathematical language. Thus, the question often arises about the references of such expressions as 0, -1, i, $\varnothing$, ω, etc. Although they seem to refer to something, we cannot directly identify their references in the way that allows everyone to agree on what they are.

An early structuralist approach to this question was made by Dedekind [5], where he defined natural numbers as any structure (a triple of a set, a function and an element) that satisfies certain conditions. However, quite different approaches to natural numbers were also taken by Cantor, Frege, Russell, and later by Zermelo, von Neuman, etc. They constructed, in one way or another, objects which behave like natural numbers out of objects or notions which they thought were at their disposal; that is, sets or properties (concepts), things which they believed were sufficiently evident. Set theory was particularly popular, and became regarded as the *foundation* of mathematics on which every branch of mathematics is to be built.

Structuralism has gained popularity in the philosophy of mathematics since Paul Benacerraf [4] raised the question about set-theoretic definitions of natural numbers: if natural numbers can be defined in several different ways in set theory, which natural numbers are intended, when we talk about them? The structuralist answer is: "Anything will equally do as long as it shares the structure essential to natural numbers."

The basic slogans of structuralism are as follows:

- Mathematical objects are only identified by relations in some structure.
- A mathematical theory is not about internal properties of individual objects, but about relations in the structure expressed by the theory.
- A theorem of a theory states some fact about the common structure of arbitrary models that satisfy the theory.

Structuralism saves us the trouble of thinking about the essence of mathematical objects —— what they really are, etc. Moreover it is obvious that present-day mathematicians treat mathematical objects structurally. On the other hand, to be a structuralist means that one inevitably undertakes the task of explaining what a mathematical structure is. Some philosophers like Shapiro and Hellman explain the notion of structure basically in terms of model theory. However this seems to bring about no less complicated problems than their earlier ones: problems of the existence of, the reference to, the underdeterminacy of a structure, etc.

Meanwhile, in mathematics, with the development of category theory purely structural approaches to mathematics flourished. Some see this movement as "revolution" that will turn over the dominance of set theory as foundational framework for mathematics.[1] However, despite the great success of category theory, philosophers of mathematics do not seem to pay much attention to it.

It is thus that Awodey [1] suggests that philosophers of mathematics use category theory if they want to approach mathematics structurally. The reasons for his suggestion are: that the notion of mathematical structure is better clarified by category theory than by set theory, or any other known method; that using category theory means doing everything structurally; and that category theory makes it possible to treat a variety of different mathematical structures and notions in a uniform way, making it easier to see connections among them. In [1], Awodey explains how familier notions in set theory and logic are derived and connected within topos theory.

However, Hellman [7] objects to the suggestion in favor of his modal structuralism. Here we will consider two of the questions he raises. One is about what the assertory axioms are. Hellman distinguishes two kinds of axioms: one is "formal" or "schematic" and plays only the role of definition, while the other is "assertory" i. e. asserts some "truth" about something. He thinks that a sufficient account for structuralism must provide a set of assertoric axioms. For example, set theory has one, while group theory does not. A set-theoretic axiom, say, "if $x \in A$ and $x \in B$ then $x \in A \cap B$," expresses some truth about concrete objects, namely sets, while a group-theoretic axiom such as "$(xy)z = x(yz)$ for all $x, y, z \in G$" only constitutes a part of the definition of what a group is. In particular, the axioms of category theory are of this character, and tell us nothing true of any concrete structure.

Related to the above question is the question about "home address" or about the mathematical existence. Hellman asks "where do categories come from and where do they live?" (p. 136) He says that category theory simply does not address this question. Thus he concludes that category theory is insufficient as a framework for structuralism. Behind this diagnosis, there is a belief that a sufficient account for structuralism must answer the following questions[2]:

1. what primitive notions and background logic are employed?

2. what are the assertory axioms?

[1] Cf. Goldblatt [6], §1.3.

[2] Cf. Hellman [8], pp. 537f.

3. are stuructures-as-objects eliminated?

4. how are the mathematical existence of structures and their indefinite extendability explained?

5. how is reference to structures, or epistemic access to structures possible?

which category theory hardly does.

Awodey [2] does not deny Hellman in that category theory does not address these questions, but retorts that it is the very nature of mathematics: the "schematic" and "top-down" character. Mathematics has nothing to say about any concrete object in any absolute fashion, but only in a hypothetical and indeterminate fashion. In other word, mathematical statements are always of the form "if $A(X)$ then $B(X)$" leaving X unspecified and even unquantified. In addition, axioms of mathematics are given from top down rather than from bottom up. Take group theory for example. We do not have the whole universe of groups to begin group theory with, and of which its axioms and theorems are true. Instead, the axioms give a condition under which anything is qualified as group. And this is the way mathematical practice is usually carried out.

Of course Hellman would not deny the schematic and top-down character in mathematical practice. What he would deny is perhaps the idea of mathematical language being totally ungrounded, so to speak. This is understandable, since we are accustomed to the time-honored principle called compositionality —— or the "bottom-up" principle: the meaning of a complex expression is determined by the meanings of its components and the configuration in which they are arranged. Therefore, in order for a statement to be meaningful, each of its components should be meaningful to begin with. This principle seems too obvious to doubt. Even when a mathematical statement seems to contain indeterminate components, there must be something they "really" mean. So the task of the philosophy of mathematics is to find out that something, and get mathematics grounded on it.

This perhaps is what is at stake in the controversy.

3 The obstacle to reconciliation

We are long accustomed to assuming that a meaningful expression should not leave any of its component indeterminate. In other words, if an expression contains an indeterminate component, then so is the meaning of the whole expression. In particular, if the expression is a proposition, its truth-value is indeterminate. But a proposition without definite truth-value is not a proposition at all!

This line of thought is affected by the concern for "having a homogeneous semantical theory in which semantics for the propositions of mathematics parallel the semantics for the rest of the language" (Benacerraf [3], p. 403). Benacerraf considers this concern to be one of the two main motivations for the account of mathematical truth.[3] But he also confesses that he is 'indulging here the fiction that we *have* semantics for "the rest of language, " or, more precisely, that the proponents of the views that take their impetus from this concern often think of themselves as having such semantics, at least for philosophically important segments of the language' (p. 403n). In addition, Benacerraf says he assumes that "truth conditions for the language (e. g., English) to which mathematese appears to belong are to be elaborated much along the lines that Tarski articulated" (p. 410). No doubt "philosophically important segments of the language" in his fiction is such that we can apply a usual analysis of terms, function symbols, predicates, logical connectives and quantifiers to them. He calls such language "referential." This assumption is not unique to Benacerraf, but quite common to philosophers of mathematics including Hellman.

I will call "the referentiality assumption" the assumption that mathematical language must be interpreted referential as our ordinary language is. It is this assumption (or "fiction") that I think prevents philosophers like Hellman from appreciating Awodey's suggestion. This assumption seems to be so wrong as to be harmful at least in two respects. For one thing, it disguises the fact that we have no homogeneous semantical theory even for our ordinary language. In fact, a substantial proportion of our ordinary language is not referential. I am doubtful whether it is possible to dismiss it as philosophically unimportant. For another, it is not obvious that mathematical language must be treated as referential. I will explain these in the next section.

4 How to get rid of the obstacle

In the previous section, we saw that the referentiality assumption is an obstacle in the way of Hellman accepting categorical structuralism. Thus, we shall try to refute it in this section.

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[3] The other is "the concern that the account of mathematical truth mesh with a reasonable epistemology" (*ibid.*).

4.1 Are non-referential segments of language negligible?

What is in the non-referential part of language? We sometimes use language to establish some institutional facts. Such usage is called "performative," and indeed paid much attention to by philosophers of language. We name persons, things and notions; we make promises, rules and laws; and in particular, we define new terms.

In these cases we use sentences of which it is useless to ask whether they are true or not, or what is the reference of each of their components. For example, consider a sign that reads, "If you park here, you are fined $50." The function of this sentence is not to report some information about something, but to set out a rule that may be applied anything that satisfies a certain condition. What truth-value or truth-condition does this sentence have? What is the reference of "you" here? These questions do not seem to be sensible to ask.

Focusing on definition, the role of definition is multifold. Sometimes definition serves as a convenient abbreviation to refer to something. For example, we use "a brother of mine" instead of saying "a child of a parent of mine who is male and is different from me." Sometimes a definition makes clear a subtle idea that has been unnamed but only vaguely perceived: in the way Horace Walpole invented the word "serendipity." Sometimes we use definition to introduce a totally new idea by combining ideas that is already known. For example, I define the word "chilrent" as a person who is both a child and a parent of another person. More precisely:

> x is called a *chilrent of* y if and only if x is a child of y and x is a parent of y.

Then we can use this word and deduce a lot of statements about chilrenthood.

What is important here is that before my definition, the string "chilrent" has no meaning at all, but it has now become a fully meaningful expression that we can use in our ordinary discourse.[4] I can ask one, "Do you have a chilrent?", and one will reply "No." The pragmatic concern set aside, this coversation is as meaningful as when I ask one if one has a cousin. In short, definition is a performative speech act by which a term acquires a new definite meaning for our later use. It makes it an institutional fact that the term has that particular meaning.

We use language in order to extend or modify language and the environment in which sentences are evaluated. This is a striking feature of human language. And I

[4] In many cases in mathematics, definition gives a new specific meaning to an already existent term.

do not regard it to be philosophically unimportant. But if the word "philosophically important segments" here is to mean, in the first place, something worth the attention of philosophers of mathematics, we should go on to the next question.

4.2 Must mathematical language be viewed as referential?

In the previous subsection, we saw examples of non-referential segments of language, and among them is definition. Definition deserves a special attention from us because it is part of mathematical activity. In fact, definition plays a central role in mathematics. We simply cannot start mathematics without definition.

As a performative speech act, definition makes an institutional fact, and its truth is not to be questioned. Hellman argues that mathematics must have some axioms that are assertory in its foundation. We do not try to refute his argument directly, but see how if we interpret mathematics otherwise, that is, think of all axioms in mathematics as merely definitional.

Once we have done away with the idea that some axioms are true of something, no question will arise if axioms in mathematics contain indeterminate components. As we saw, it is natural that rules contain indeterminate component. When we see the sign that reads "if you park here, you are fined $50," we do not wonder who this "you" is. Nor do we think that this sentence is quantified, because quantification needs a fixed range and perhaps that is not intended by this rule. The same holds of definition. When I defined the word "chilrent" neither a particular person nor a fixed range was intended. What is more, it does not matter that there exists no one who satisfies the condition for being chilrent. By this fact, Hellman and some others would worry that everything can be said of chilrenthood. This happens only when we interpret the "if" part of the definition truth-functionally. We will discuss this later.

Seeing axioms as definitional makes the situation simple. At least three of five questions Hellman listed —— 2, 4 and 5 —— become irrelevant. This is substantial gain. What then is the loss? Some may become at a loss what mathematics is all about. Others may dislike the idea that mathematical truth is all by definition. However they have not escaped from the referentiality assumption.

Awodey, trying to show difference between his position and old-fashioned "if-thenism," says:

> The truth of the consequent statement doesn't depend on some unknown or unknowable antecedent conditions; rather it *applies* only to those cases specified by the antecedent description. ([2], p. 9)

Thus, he does not interpret if-then truth-functionally. The antecedent works to pick up those cases to which the consequent can be applied. If-then in mathematics expresses *applicability condition*, so to speak. One may be reminded of the Brouwer-Heyting-Kolmogorov interpretation of if-then. In the BHK interpretation, "if A then B" means "there is a procedure p that turns any proof x of A into a proof $p(x)$ of B." Thus, p is applied only to proofs of A to produce proofs of B, and not anything else. Note that one need not be an intuitionist or a constructivist to employ the BHK interpretation of if-then, because a proof x of A or a procedure p can be classical.

Again, this usage of if-then is not marginal in "the rest of language." If-then that appears in rules, promises, laws and so on is of this character. One may also recall that in a programming language, if-then is usually not a Boolean expression but a command in a special form. For example, the expression **if** $x \neq 0$ **then** $y := y/x$ **end** tells us to let y be y/x if x is not 0, and do nothing otherwise. This is also a good example of an exception to the principle of compositionality. If x is 0, then the expression y/x is meaningless. However, the whole expression is still evaluated normally, and the interpreter does nothing, as expected.

To sum up this section, non-referential segments of language is worth our consideration, to explain mathematics in non-referential terms has several advantages, and even then continuity with some part of ordinary language is still maintained.

5 conclusion

Given what impact category theory has had on mathematics, logic and computer science, it is rather surprising that structuralists such as Shapiro, Resnik and Hellman pay scant attention to category theory. Reviewing the debate between Awodey and Hellman, we saw that the reason for the scantness can be trace back to the traditional concern for having homogenious semantical theory for mathematics and the rest of the language. Behind this concern lie the referentiality assumption that both our ordinary language and mathematical language are referential. Awodey's reply to Hellman, emphasizing the schematic and top-down character of mathematical practice, casts doubt on this assumption. Thus we tried to show that non-referential aspects of language is as important, and that viewing mathematics as non-referential has several advantages.

References

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